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Representations of coherent states in non-orthogonal bases

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Abstract
Starting with the canonical coherent states, we demonstrate that all the so-called nonlinear coherent states, used in the physical literature, as well as large classes of other generalized coherent states, can be obtained by changes of bases in the underlying Hilbert space. This observation leads to an interesting duality between pairs of generalized coherent states, bringing into play a Gelfand triple of (rigged) Hilbert spaces. Moreover, it is shown that in each dual pair of families of nonlinear coherent states, at least one family is related to a (generally) non-unitary projective representation of the Weyl–Heisenberg group, which can then be thought of as characterizing the dual pair.

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1. Introduction

We begin with the well-known *canonical coherent states* (CCS), $|z\rangle$. In the physical literature (see, e.g., Ali *et al* 2000, Klauder and Skagerstam 1985, Perelomov 1986), these are written in terms of the so-called Fock basis $|n\rangle$, $n = 0, 1, 2, \ldots, \infty$ (or *number states*):

$$|z\rangle = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \qquad \forall z \in \mathbb{C} \tag{1.1}$$

where the normalization constant, $\mathcal{N}(|z|^2) = e^{z^2}$, is chosen so as to ensure that $\langle z|z\rangle = 1$. The basis vectors $|n\rangle$ are orthonormal in the underlying Hilbert space, often termed a *Fock space*. However, in this paper we shall use a somewhat more general notation and write

$$|z\rangle = \eta_z = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n \qquad \forall z \in \mathbb{C} \tag{1.2}$$

defined as vectors in an abstract (complex, separable) Hilbert space $\mathfrak{H}$, for which the vectors

ϕ_n form an orthonormal basis:

$$\langle \phi_n | \phi_m \rangle_{\mathfrak{H}} = \delta_{nm} \qquad n, m = 0, 1, 2, \ldots, \infty. \tag{1.3}$$

The so-called *nonlinear* coherent states are then defined (see, e.g., Man'ko *et al* 1997) by replacing the $n!$ in the denominator following the summation sign in (1.2) by $x_n! := x_1 x_2 x_3 \cdots x_n$, where $x_1, x_2, x_3, \ldots,$ is a sequence of nonzero positive numbers and, by convention, $x_0! = 1$. Thus, one obtains the vectors

$$\eta_z^{\mathrm{nl}} = \mathcal{N}_{\mathrm{nl}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{x_n!}} \phi_n \tag{1.4}$$

where again $\mathcal{N}_{\mathrm{nl}}(|z|^2)$ is an appropriate normalizing constant. Of course, these are only defined for $z \in \mathcal{D}$, where $\mathcal{D}$ is the open domain in the complex plane defined by $|z| < L$, with $L^2 = \lim_{n\to\infty} x_n$ (provided, of course, that this limit exists and is nonzero). It is our intention to prove in this paper that such a family of nonlinear coherent states can be obtained via a linear transformation on the Hilbert space $\mathfrak{H}$, which will amount to replacing the orthonormal set $\{\phi_n\}_{n=0}^{\infty}$ by, in general, a non-orthogonal basis. Under appropriate restrictions, the inverse transformation leads to a *dual* family of nonlinear coherent states. This duality is related to a Gelfand triple (Gelfand and Vilenkin 1964) of (rigged) Hilbert spaces. Furthermore, just as the canonical coherent states (1.2) can also be defined as the orbit of a single vector under a projective, unitary representation of the Weyl–Heisenberg group:

$$\eta_z = D(z)\phi_0 \qquad D(z) = \mathrm{e}^{\overline{z}a - za^{\dagger}} \tag{1.5}$$

it will emerge that in a dual pair of families of nonlinear coherent states, at least one family is the orbit of a projective, *non-unitary* representation of this same group. It will also be demonstrated, in particular, that the well-known *photon-added* states (Agarwal and Tara 1991, Roy and Mehta 1995) and the *binomial states* (Fu *et al* 2000) can also be obtained by such a linear transformation on $\mathfrak{H}$. However, in these two cases, the nonlinear coherent states constructed using the resulting non-orthogonal bases, again turn out to be canonical coherent states and indeed, it is possible to characterize a fairly general class of transformations under which such a situation prevails.

It ought to be mentioned at this point that the fact that nonlinear coherent states are related to a choice of a new scalar product on the Hilbert space, has been observed before (Beckers *et al* 2001, Man'ko *et al* 1997). Similarly, the existence of a generalized displacement like operator, related to nonlinear coherent states, has been studied earlier (Roy and Roy 2000). However, we unify all these concepts by a systematic application of a certain class of linear transformations on the underlying Hilbert space. The resultant appearance of a duality among families of nonlinear coherent states and of a Gelfand triple in this context, as well as the connection with non-unitary representations of the Weyl–Heisenberg group, has apparently not been noticed before.

2. The general setting

The primary object for this discussion will be an abstract Hilbert space $\mathfrak{H}$. Let T be an operator on this space with the properties

(1) T is densely defined and closed; we denote its domain by $\mathcal{D}(T)$.
(2) T^{-1} exists and is densely defined, with domain $\mathcal{D}(T^{-1})$.
(3) The vectors $\phi_n \in \mathcal{D}(T) \cap \mathcal{D}(T^{-1})$ for all n and there exist non-empty open sets $\mathcal{D}_T$ and $\mathcal{D}_{T^{-1}}$ in $\mathbb{C}$ such that $\eta_z \in \mathcal{D}(T), \forall z \in \mathcal{D}_T$ and $\eta_z \in \mathcal{D}(T^{-1}), \forall z \in \mathcal{D}_{T^{-1}}$.

Note that condition (1) implies that the operator $T^*T = F$ is self-adjoint.

Let

$$\phi_n^F := T^{-1}\phi_n \qquad \phi_n^{F^{-1}} := T\phi_n \qquad n = 0, 1, 2, \ldots, \infty \tag{2.1}$$

we define the two new Hilbert spaces:

(1) $\mathfrak{H}_F$, which is the completion of the set $\mathcal{D}(T)$ in the scalar product

$$\langle f|g\rangle_F = \langle f|T^*Tg\rangle_{\mathfrak{H}} = \langle f|Fg\rangle_{\mathfrak{H}}. \tag{2.2}$$

The set $\{\phi_n^F\}$ is orthonormal in $\mathfrak{H}_F$ and the map $\phi \longmapsto T^{-1}\phi, \phi \in \mathcal{D}(T^{-1})$ extends to a *unitary* map between $\mathfrak{H}$ and $\mathfrak{H}_F$. If both T and T^{-1} are bounded, $\mathfrak{H}_F$ coincides with $\mathfrak{H}$ as a set. If T^{-1} is bounded, but T is unbounded, so that the spectrum of F is bounded away from zero, then $\mathfrak{H}_F$ coincides with $\mathcal{D}(T)$ as a set.

(2) $\mathfrak{H}_{F^{-1}}$, which is the completion of $\mathcal{D}(T^{*-1})$ in the scalar product

$$\langle f|g\rangle_{F^{-1}} = \langle f|T^{-1}T^{*-1}g\rangle_{\mathfrak{H}} = \langle f|F^{-1}g\rangle_{\mathfrak{H}}. \tag{2.3}$$

The set $\{\phi_n^{F^{-1}}\}$ is orthonormal in $\mathfrak{H}_{F^{-1}}$ and the map $\phi \longmapsto T\phi, \phi \in \mathcal{D}(T)$ extends to a *unitary* map between $\mathfrak{H}$ and $\mathfrak{H}_{F^{-1}}$. If the spectrum of F is bounded away from zero; then F^{-1} is bounded and one has the inclusions

$$\mathfrak{H}_F \subset \mathfrak{H} \subset \mathfrak{H}_{F^{-1}}. \tag{2.4}$$

We shall refer to the spaces $\mathfrak{H}_F$ and $\mathfrak{H}_{F^{-1}}$ as a *dual pair* and when (2.4) is satisfied, the three spaces $\mathfrak{H}_F, \mathfrak{H}$ and $\mathfrak{H}_{F^{-1}}$ will be called a *Gelfand triple* (Gelfand and Vilenkin 1964). (Actually, this is a rather simple example of a Gelfand triple, consisting only of a triplet of Hilbert spaces (Antoine *et al* 2002).)

Let B be a (densely defined) operator on $\mathfrak{H}$ and $B^\dagger$ its adjoint on this Hilbert space. Assume that $\mathcal{D}(B) \subset \mathcal{D}(F)$. Then unless $[B, F] = 0$, the adjoint of B, considered as an operator on $\mathfrak{H}_F$ and which we denote by B_F^*, is different from $B^\dagger$. Indeed,

$$\begin{aligned}\langle f|Bg\rangle_F &= \langle f|FBg\rangle_{\mathfrak{H}} = \langle B^\dagger Ff|g\rangle_{\mathfrak{H}} = \langle FF^{-1}B^\dagger Ff|g\rangle_{\mathfrak{H}}\\ &= \langle F^{-1}B^\dagger Ff|Fg\rangle_{\mathfrak{H}} = \langle F^{-1}B^\dagger Ff|g\rangle_F \qquad \forall f, g \in \mathcal{D}(F).\end{aligned}$$

Thus

$$B_F^* = F^{-1}B^\dagger F.$$

On $\mathfrak{H}$ we take the operators $a, a^\dagger, N = a^\dagger a$:

$$a\phi_n = \sqrt{n}\phi_{n-1} \qquad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1} \qquad N\phi_n = n\phi_n. \tag{2.5}$$

These operators satisfy

$$[a, a^\dagger] = 1 \qquad [a, N] = a \qquad [a^\dagger, N] = -a^\dagger. \tag{2.6}$$

On $\mathfrak{H}_F$ we have the transformed operators:

$$a_F = T^{-1}aT \qquad a_F^\dagger = T^{-1}a^\dagger T \qquad N_F = T^{-1}NT. \tag{2.7}$$

These operators satisfy the same commutation relations as $a, a^\dagger$ and N:

$$\left[a_F, a_F^\dagger\right] = 1 \qquad [a_F, N_F] = a_F \qquad \left[a_F^\dagger, N_F\right] = -a_F^\dagger. \tag{2.8}$$

Also on $\mathfrak{H}_F$

$$a_F\phi_n^F = \sqrt{n}\phi_{n-1} \qquad a_F^\dagger\phi_n^F = \sqrt{n+1}\phi_{n+1}^F \qquad N_F\phi_n^F = n\phi_n^F. \tag{2.9}$$

Clearly, considered as operators on $\mathfrak{H}_F$, a_F and $a_F^\dagger$ are adjoints of each other and indeed they are just the unitary transforms on $\mathfrak{H}_F$ of the operators a and $a^\dagger$ on $\mathfrak{H}$. On the other hand, if we take the operator a_F, let it act on $\mathfrak{H}$ and look for its adjoint on $\mathfrak{H}$ under this action, we obtain by (2.7) the operator $a^{\sharp} = T^{*}a^{\dagger}T^{*-1}$ which, in general, is different from $a_F^{\dagger}$ and also $[a_F, a^{\sharp}] \neq I$, in general. In an analogous manner, we shall define the corresponding operators $a_{F^{-1}}, a_{F^{-1}}^{\dagger}$, etc, on $\mathfrak{H}_{F^{-1}}$.

We thus obtain three unitarily equivalent sets of operators: $a, a^{\dagger}, N$, defined on $\mathfrak{H}$, a_F, $a_F^{\dagger}$, N_F, defined on $\mathfrak{H}_F$ and $a_{F^{-1}}, a_{F^{-1}}^{\dagger}, N_{F^{-1}}$ defined on $\mathfrak{H}_{F^{-1}}$. On their respective Hilbert spaces, they define under commutation the standard oscillator Lie algebra. On the other hand, if they are all considered as operators on $\mathfrak{H}$, the algebra generated by them and their adjoints on $\mathfrak{H}$ (under commutation) is, in general, very different from the oscillator algebra and could even be an infinite dimensional Lie algebra.

Writing $A = a_F$, $A^{\dagger} = a^{\sharp}$, both considered as operators on $\mathfrak{H}$, if they satisfy the relation

$$AA^{\dagger} - \lambda A^{\dagger}A = C(N) \tag{2.10}$$

where $\lambda \in \mathbb{R}_*^+$ is a constant and $C(N)$ is a function of the operator N, then the three operators A, $A^{\dagger}$, $H = \frac{1}{2}(AA^{\dagger} + A^{\dagger}A)$ are said to generate a *generalized oscillator algebra* or *deformed oscillator algebra* (Borzov *et al* 1997). Note that on $\mathfrak{H}$, A and $A^{\dagger}$ are adjoints of each other.

3. Construction of coherent states

Consider the vectors

$$\eta_z^F = T^{-1}\eta_z = \mathcal{N}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\phi_n^F \tag{3.1}$$

on $\mathfrak{H}_F$. These are the images of the η_z in $\mathfrak{H}_F$ and are the normalized canonical coherent states on this Hilbert space (recall that the vectors ϕ_n^F are orthonormal in $\mathfrak{H}_F$). Similarly, define the vectors

$$\eta_z^{F^{-1}} = T\eta_z = \mathcal{N}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\phi_n^{F^{-1}} \tag{3.2}$$

as the CCS η_z unitarily transported from $\mathfrak{H}$ to $\mathfrak{H}_{F^{-1}}$.

We would now like to consider the η_z^F as being vectors in $\mathfrak{H}$ and similarly the vectors $\eta_z^{F^{-1}}$ also as vectors in $\mathfrak{H}$. To what extent can we then call them (generalized) coherent states? Specifically, we would like to find an orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$ in $\mathfrak{H}$ and a transformation $w = f(z)$ of the complex plane to itself such that:

(a) we could write

$$\eta_z^F = \zeta_w = \mathcal{N}'(|w|^2)^{-1/2}\Omega(w)\sum_{n=0}^{\infty}\frac{w^n}{\sqrt{[x_n!]}}\psi_n \tag{3.3}$$

where $\mathcal{N}'$ is a new normalization constant, $\Omega(w)$ is a phase factor and $\{x_n\}_{n=1}^{\infty}$ is a sequence of nonzero positive numbers, to be determined;

(b) there should exist a measure $\mathrm{d}\lambda(\rho)$ on $\mathbb{R}^+$, such that with respect to the measure $\mathrm{d}\mu(w,\overline{w}) = \mathrm{d}\lambda(\rho)\,\mathrm{d}\vartheta$ (where $w = \rho\mathrm{e}^{\mathrm{i}\vartheta}$) the resolution of the identity,

$$\int_{\mathcal{D}} |\zeta_w\rangle\langle\zeta_w|\mathcal{N}'(|w|^2)\,\mathrm{d}\mu(w,\overline{w}) = I \tag{3.4}$$

would hold on $\mathfrak{H}$ (as is the case with the canonical coherent states). Here again, $\mathcal{D}$ is the domain of the complex plane, $\mathcal{D} = \{w \in \mathbb{C} \mid |w| < L\}$, where $L^2 = \lim_{n\to\infty} x_n$.

A general answer to the above question may be hard to find. But we present below several classes of examples, all physically motivated, for which the above construction can be carried out. These include in particular all the so-called nonlinear, deformed and squeezed coherent states, which appear so abundantly in the quantum optical and physical literature (see, for example, Man'ko *et al* 1997, Odzijewicz 1998, Simon *et al* 1988).

Whenever the two sets of vectors $\{\eta_z^F\}$ and $\{\eta_z^{F^{-1}}\}$ form coherent state families in the above sense, we shall call them a *dual pair*.

4. Examples of the general construction

4.1. Example 1. Photon-added and binomial states as bases

Let T be an operator such that T^{-1} has the form

$$T^{-1} = \mathrm{e}^{\lambda a^\dagger} G(a) \tag{4.1}$$

where $\lambda \in \mathbb{R}$ and $G(a)$ is a function of the operator a such that T and T^{-1} satisfy the postulated conditions (1)–(3) of section 2. (The operator $G(a)$ could, for example, be defined by taking an entire analytic function $G(z)$ with real coefficients and nonzero in the finite plane, and then setting $G(a)\eta_z = G(z)\eta_z$ for all $z \in \mathbb{C}$.) It is easily verified that

$$\mathrm{e}^{\lambda a^\dagger} a = (a - \lambda I)\, \mathrm{e}^{\lambda a^\dagger} \qquad \mathrm{e}^{\lambda a} a^\dagger = (a^\dagger + \lambda I)\, \mathrm{e}^{\lambda a} \tag{4.2}$$

so that

$$\mathrm{e}^{\lambda a^\dagger} G(a) = G(a - \lambda I)\, \mathrm{e}^{\lambda a^\dagger}. \tag{4.3}$$

From this we compute the two transformed operators a_F and $a_F^\dagger$ on $\mathfrak{H}_F$ ($F = T^*T = \mathrm{e}^{-\lambda a} G(a^\dagger)^{-1} G(a)^{-1}\, \mathrm{e}^{-\lambda a^\dagger}$) to be

$$a_F = T^{-1} a T = a - \lambda I \qquad a_F^\dagger = T^{-1} a^\dagger T = G(a - \lambda I) a^\dagger G(a - \lambda I)^{-1}. \tag{4.4}$$

Thus, since a commutes with $G(a - \lambda I)$, we obtain

$$\left[a_F, a_F^\dagger\right] = G(a - \lambda I)[a, a^\dagger] G(a - \lambda I)^{-1} = I$$

as expected. The two operators $A = a_F$ and $A^\dagger = T^* a^\dagger T^{*^{-1}}$, defined on $\mathfrak{H}$, are

$$A = a - \lambda I \qquad A^\dagger = a^\dagger - \lambda I \tag{4.5}$$

which of course are adjoints of each other. Moreover, in this case

$$[A, A^\dagger] = I \tag{4.6}$$

so that the oscillator algebra remains unchanged.

Since, by (4.2),

$$a\, \mathrm{e}^{-\lambda a^\dagger} = \mathrm{e}^{-\lambda a^\dagger} (a - \lambda I)$$

we see that

$$T = G(a)^{-1}\, \mathrm{e}^{-\lambda a^\dagger} = \mathrm{e}^{-\lambda a^\dagger} G(a - \lambda I)^{-1}. \tag{4.7}$$

Thus we obtain the corresponding operators

$$a_{F^{-1}} = T a T^{-1} = a + \lambda I \qquad a_{F^{-1}}^\dagger = T a^\dagger T^{-1} = G(a)^{-1} a^\dagger G(a) \tag{4.8}$$

on $\mathfrak{H}_{F^{-1}}$. Once again we obtain $\left[a_{F^{-1}}, a_{F^{-1}}^\dagger\right] = I$ and similarly for the operator $A' = a_{F^{-1}} = a + \lambda I$ and its adjoint $A'^\dagger = a^\dagger + \lambda I$ on $\mathfrak{H}$.

We now define the vectors

$$\phi_n^F = T^{-1}\phi_n = \mathrm{e}^{\lambda a^\dagger} G(a) \phi_n \tag{4.9}$$

which form an orthonormal set in $\mathfrak{H}_F$, and build the corresponding canonical coherent states

$$\eta_z^F = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n^F = \mathrm{e}^{\lambda a^\dagger} G(a) \eta_z \tag{4.10}$$

on $\mathfrak{H}_F$. Considering these as vectors in $\mathfrak{H}$, and taking account of the fact that

$$a\,\mathrm{e}^{\lambda a^\dagger} G(a) = \mathrm{e}^{\lambda a^\dagger} G(a)(a + \lambda I)$$

we see that

$$a\eta_z^F = (z+\lambda)\eta_z^F. \tag{4.11}$$

Thus, up to a constant factor, η_z^F is just the canonical coherent state on $\mathfrak{H}$ corresponding to the point $(z+\lambda) \in \mathbb{C}$ (note that since the canonical coherent states can be obtained as solutions to a first-order differential equation, $(x + \mathrm{d}/\mathrm{d}x)\eta_z = z\eta_z$, the solution is unique, up to a constant, for each $z \in \mathbb{C}$, i.e., to each $z \in \mathbb{C}$, there corresponds exactly one vector η such that $a\eta = z\eta$). We write, therefore,

$$\eta_z^F = C(\lambda, z) \sum_{n=0}^{\infty} \frac{(z+\lambda)^n}{\sqrt{n!}} \phi_n$$

where the constant $C(\lambda, z)$ can be computed by going back to (4.10). Indeed, we have

$$\begin{aligned}
\eta_z^F &= \mathrm{e}^{\lambda a^\dagger} G(a)\eta_z = G(z)\,\mathrm{e}^{\lambda a^\dagger} \eta_z \\
&= G(z)\,\mathrm{e}^{-\frac{|z|^2}{2}}\,\mathrm{e}^{\lambda a^\dagger}\,\mathrm{e}^{z a^\dagger} \phi_0 = G(z)\,\mathrm{e}^{-\frac{|z|^2}{2}}\,\mathrm{e}^{\frac{|z+\lambda|^2}{2}} \eta_{z+\lambda} \\
&= G(z)\,\mathrm{e}^{\lambda(\mathrm{Re}(z)+\frac{\lambda}{2})} \eta_{z+\lambda}.
\end{aligned}$$

Thus, we obtain $C(\lambda, z) = G(z)\,\mathrm{e}^{-\frac{|z|^2}{2}}$ and

$$\eta_z^F = G(z)\,\mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(z+\lambda)^n}{\sqrt{n!}} \phi_n = G(z)\,\mathrm{e}^{\lambda(\mathrm{Re}(z)+\frac{\lambda}{2})} \eta_{z+\lambda}. \tag{4.12}$$

Comparing (4.12) with (3.3) and writing $\eta_z^F = \zeta_{z+\lambda}$, we find that $w = z + \lambda$, $x_n = n$ and $\psi_n = \phi_n$. Furthermore, $\mathcal{N}'(|w|^2) = \mathrm{e}^{|z|^2}|G(z)|^{-2}$ and $\Omega(w) = \mathrm{e}^{\mathrm{i}\Theta(w)}$, where we have written $G(z) = |G(z)|\,\mathrm{e}^{\mathrm{i}\Theta(w)}$. It is remarkable that in this example while η_z^F is written in (4.10) in terms of a non-orthonormal basis $\{\phi_n^F\}_{n=0}^{\infty}$, when these vectors are considered as constituting a basis for $\mathfrak{H}$, its transcription in terms of the orthonormal basis $\{\phi_n\}_{n=0}^{\infty}$ only involves a shift in the variable z and no change in the components.

It is now straightforward to write down a resolution of identity, following the pattern of the canonical coherent states. Indeed, writing $w = z + \lambda = \rho\,\mathrm{e}^{\mathrm{i}\theta}$, we have (on $\mathfrak{H}$)

$$\int\!\!\int_{\mathbb{C}} |\zeta_w\rangle\langle\zeta_w| \mathcal{N}'(|w|^2)\,\mathrm{d}\mu(w,\overline{w}) = I \qquad \mathrm{d}\mu(w,\overline{w}) = \frac{\mathrm{e}^{-\rho^2}}{\pi}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{4.13}$$

The dual CS $\eta_z^{F^{-1}}$ are obtained by replacing the ϕ_n^F in (4.9) by $\phi_n^{F^{-1}} = T\phi_n = G(a)^{-1}\,\mathrm{e}^{-\lambda a^\dagger}\phi_n$. But since $G(a)^{-1}\,\mathrm{e}^{-\lambda a^\dagger} = \mathrm{e}^{-\lambda a^\dagger} G(a - \lambda I)^{-1}$, we have

$$\phi_n^{F^{-1}} = \mathrm{e}^{-\lambda a^\dagger} G(a - \lambda I)^{-1}\phi_n. \tag{4.14}$$

Hence, using the same argument as with the ϕ_n^F, we arrive at

$$\eta_z^{F^{-1}} = G(z-\lambda)^{-1}\,\mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(z-\lambda)^n}{\sqrt{n!}}\phi_n = G(z-\lambda)^{-1}\,\mathrm{e}^{-\lambda(\mathrm{Re}(z)-\frac{\lambda}{2})}\eta_{z-\lambda}. \tag{4.15}$$

Thus, in the present case (up to normalization), the dual pair of states η_z^F and $\eta_z^{F^{-1}}$ is obtained simply by replacing λ by $-\lambda$.

It is clear now that the above construction can be carried out for any operator T^{-1} which satisfies the commutation relation

$$[a, T^{-1}] = \lambda T^{-1} \qquad \lambda \in \mathbb{R} \tag{4.16}$$

with a.

Two particular cases of the operator T^{-1} in (4.1) are of special interest. In the first instance take $G(a) = I$, so that $T^{-1} = \mathrm{e}^{\lambda a^\dagger}$. The vectors $\phi_n^F = T^{-1}\phi_n$ may easily be calculated. Indeed we obtain

$$\phi_n^F = \sum_{k=0}^{\infty} \frac{(\lambda a^\dagger)^k}{k!}\phi_n = \mathrm{e}^{\frac{\lambda^2}{2}} \frac{a^{\dagger n}}{\sqrt{n!}}\eta_\lambda \tag{4.17}$$

which (up to normalization) are the well-known *photon-added coherent states* of quantum optics (Agarwal and Tara 1991, Roy and Mehta 1995). Hence in this case we write $\phi_n^F = \phi_{\lambda,n}^{\mathrm{pa}}$. We denote the corresponding coherent states by $\eta_{\lambda,z}^{\mathrm{pa}}$ and note that

$$\begin{aligned}\eta_z^F := \eta_{\lambda,z}^{\mathrm{pa}} &= \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\phi_{\lambda,n}^{\mathrm{pa}} = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \mathrm{e}^{\frac{\lambda^2}{2}} \mathrm{e}^{za^\dagger}\eta_\lambda \\ &= \mathrm{e}^{\lambda(x+\frac{\lambda}{2})}\eta_{z+\lambda}\end{aligned} \tag{4.18}$$

where $x = \mathrm{Re}(z)$. Clearly if $\lambda \longrightarrow 0$, then $\eta_{\lambda,z}^{\mathrm{pa}} \longrightarrow \eta_z$. It ought to be emphasized at this point, however, that while the vectors $\phi_{\lambda,n}^{\mathrm{pa}}$ are (up to normalization) photon-added coherent states, the vectors $\eta_{\lambda,z}^{\mathrm{pa}}$ are just (up to normalization) canonical coherent states. The dual set of coherent states, $\eta_z^{F^{-1}}$, is obtained by replacing λ by $-\lambda$ so that the states $\eta_{\lambda,z}^{\mathrm{pa}}$ and $\eta_{-\lambda,z}^{\mathrm{pa}}$, $z \in \mathbb{C}$, are in duality, and we have the interesting relation

$$\left\langle \eta_{-\lambda,z}^{\mathrm{pa}} \middle| \eta_{\lambda,z}^{\mathrm{pa}} \right\rangle_{\mathfrak{H}} = \mathrm{e}^{-\lambda(\lambda+2\,\mathrm{i}y)}. \tag{4.19}$$

On $\mathfrak{H}_F$ we have the creation and annihilation operators (see (4.4)),

$$a_F = a - \lambda I \qquad a_F^\dagger = a^\dagger \tag{4.20}$$

which are adjoints of each other on $\mathfrak{H}_F$, but clearly not so on $\mathfrak{H}$. However, on $\mathfrak{H}$ we have the two operators A and $A^\dagger$ as in (4.5):

$$A = a - \lambda I \qquad A^\dagger = a^\dagger - \lambda I.$$

As the second particular case of (4.1), we take $\lambda = 0$ and $G(a) = \mathrm{e}^{\mu a}$, $\mu \in \mathbb{R}$, i.e., $T^{-1} = \mathrm{e}^{\mu a}$. The basis vectors are now

$$\phi_n^F = \mathrm{e}^{\mu a}\phi_n = \sqrt{n!}\sum_{k=0}^{n} \frac{\mu^{n-k}}{\sqrt{k!}(n-k)!}\phi_n = \frac{(a^\dagger + \mu I)^n}{\sqrt{n!}}\phi_0. \tag{4.21}$$

These states have also been studied in the quantum optical literature (Fu *et al* 2000) and in view of the last expression in (4.21), we shall call them *binomial states* and write $\phi_n^F = \phi_{\mu,n}^{\mathrm{bin}}$. The coherent states, built out of these vectors as basis states, are

$$\eta_z^F := \eta_{\mu,z}^{\mathrm{bin}} = \mathrm{e}^{\mu a}\eta_z = \mathrm{e}^{-|z|^2/2}\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\phi_{\mu,n}^{\mathrm{bin}} = \mathrm{e}^{\mu x-|z|^2/2}\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\phi_n. \tag{4.22}$$

The dual CS are simply $\eta_{-\mu,z}^{\mathrm{bin}}$ and

$$\left\langle \eta_{-\mu,z}^{\mathrm{bin}} \middle| \eta_{\mu,z}^{\mathrm{bin}} \right\rangle = 1. \tag{4.23}$$

The creation and annihilation operators on $\mathfrak{H}_F$ are

$$a_F = a \qquad a_F^\dagger = a^\dagger + \lambda I \tag{4.24}$$

while the other two operators on $\mathfrak{H}$ are

$$A = a \qquad A^\dagger = a^\dagger. \tag{4.25}$$

The operators (4.24) have been studied, in the context of non-self-adjoint Hamiltonians in Beckers (1998a, 1998b, 2001). Again, it is remarkable that the coherent states $\eta_{\mu,z}^{\mathrm{bin}}$ are exactly the canonical coherent states, η_z, up to a factor.

Before leaving this example, a further point ought to be made in connection with the two basis sets $\{\phi_{\lambda,n}^{\mathrm{pa}}\}_{n=0}^\infty$ and $\{\phi_{\mu,n}^{\mathrm{bin}}\}_{n=0}^\infty$, consisting of the photon-added coherent states and the binomial states, respectively. The set $\{\phi_{\lambda,n}^{\mathrm{pa}}\}_{n=0}^\infty$ is orthonormal with respect to the operator

$$F_{\mathrm{pa}} = \mathrm{e}^{-\lambda a}\,\mathrm{e}^{-\lambda a^\dagger} = \mathrm{e}^{\lambda^2/2}\,\mathrm{e}^{-\sqrt{2}\lambda Q} \tag{4.26}$$

where

$$Q = \frac{1}{\sqrt{2}}(a + a^\dagger) \tag{4.27}$$

is the usual position operator. Thus, we have

$$\mathrm{e}^{\lambda^2/2}\langle\phi_{\lambda,n}^{\mathrm{pa}}|\mathrm{e}^{-\sqrt{2}\lambda Q}\phi_{\lambda,m}^{\mathrm{pa}}\rangle_{\mathfrak{H}} = \delta_{mn}. \tag{4.28}$$

The operator $\mathrm{e}^{-\sqrt{2}\lambda Q}$ has an absolutely continuous spectrum ranging from 0 to ∞. On the other hand, the set $\{\phi_{\mu,n}^{\mathrm{bin}}\}_{n=0}^\infty$ is orthonormal with respect to the operator

$$F_{\mathrm{bin}} = \mathrm{e}^{-\mu a^\dagger}\,\mathrm{e}^{-\mu a} = \mathrm{e}^{-\mu^2/2}\,\mathrm{e}^{-\sqrt{2}\mu Q} \tag{4.29}$$

so that

$$\mathrm{e}^{-\mu^2/2}\langle\phi_{\mu,n}^{\mathrm{bin}}|\mathrm{e}^{-\sqrt{2}\mu Q}\phi_{\mu,m}^{\mathrm{bin}}\rangle_{\mathfrak{H}} = \delta_{mn}. \tag{4.30}$$

Since for $\lambda = \mu$, $\mathrm{e}^{\lambda^2}F_{\mathrm{bin}} = F_{\mathrm{pa}}$, i.e., the two operators only differ by a constant, the vectors $\phi_{\lambda,n}^{\mathrm{pa}}$ and $\phi_{\lambda,n}^{\mathrm{bin}}$, $n = 0, 1, 2, \ldots$, must be unitarily related, up to a constant. Indeed, since in this case,

$$\phi_{\lambda,n}^{\mathrm{pa}} = \mathrm{e}^{\lambda a^\dagger}\phi_n \qquad \text{and} \qquad \phi_{\lambda,n}^{\mathrm{bin}} = \mathrm{e}^{\lambda a}\phi_n$$

we easily obtain

$$\phi_{\lambda,n}^{\mathrm{pa}} = \mathrm{e}^{\lambda^2/2}V\phi_{\lambda,n}^{\mathrm{bin}} \qquad n = 0, 1, 2, \ldots \tag{4.31}$$

where V is the unitary operator

$$V = \mathrm{e}^{-\mathrm{i}\sqrt{2}\lambda P} \qquad P = \frac{a - a^\dagger}{\mathrm{i}\sqrt{2}}. \tag{4.32}$$

Finally, note also that since $(\mathrm{e}^{-\lambda a^\dagger})^\dagger = (\mathrm{e}^{\lambda a})^{-1}$, the two sets of vectors $\{\phi_{-\lambda,n}^{\mathrm{pa}}\}_{n=0}^\infty$ and $\{\phi_{\lambda,n}^{\mathrm{bin}}\}_{n=0}^\infty$ constitute dual bases for $\mathfrak{H}$:

$$\langle\phi_{-\lambda,m}^{\mathrm{pa}}|\phi_{\lambda,n}^{\mathrm{bin}}\rangle = \delta_{mn}. \tag{4.33}$$

It should be emphasized however, that neither one of the two sets of vectors is an orthonormal basis of $\mathfrak{H}$.

4.2. Example 2. Rescaled basis states and nonlinear CS

For the next general class of examples, let the operator T^{-1} have the form

$$T^{-1} := T(N)^{-1} = \sum_{n=0}^{\infty} \frac{1}{t(n)} |\phi_n\rangle\langle\phi_n| \tag{4.34}$$

where the $t(n)$ are real numbers, having the properties:

(1) $t(0) = 1$ and $t(n) = t(n')$ if and only if $n = n'$;
(2) $0 < t(n) < \infty$;
(3) the finiteness condition for the limit

$$\lim_{n\to\infty} \left[\frac{t(n)}{t(n+1)}\right]^2 \frac{1}{n+1} = \rho < \infty \tag{4.35}$$

holds.

This last condition implies that the series

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{[t(n)]^2 n!} := S(r^2) \tag{4.36}$$

converges for all $r < L = 1/\sqrt{\rho}$. The operators T and F are now

$$T := T(N) = \sum_{n=0}^{\infty} t(n) |\phi_n\rangle\langle\phi_n| \qquad F := F(N) = \sum_{n=0}^{\infty} t(n)^2 |\phi_n\rangle\langle\phi_n|. \tag{4.37}$$

Let us define a new operator, $f(N)$, by its action on the basis vectors

$$f(N)\phi_n := \frac{t(n)}{t(n-1)}\phi_n = f(n)\phi_n \tag{4.38}$$

then

$$t(n) = f(n) f(n-1) \cdots f(1) := f(n)!. \tag{4.39}$$

Thus we have the transformed, non-orthogonal basis vectors

$$\phi_n^F = \frac{1}{t(n)}\phi_n = \frac{1}{f(n)!}\phi_n \tag{4.40}$$

so that if $\psi = \sum_{n=0}^{\infty} c_n \phi_n$ and $\psi' = \sum_{n=0}^{\infty} c'_n \phi_n$ are vectors in $\mathfrak{H}$ which lie in the domain of T^{-1}, then their scalar product in $\mathfrak{H}_F$ is

$$\langle\psi|\psi'\rangle_F = \sum_{n=0}^{\infty} \frac{\overline{c}_n c'_n}{[f(n)!]^2}.$$

We shall call the vectors (4.40) *rescaled basis states*.

The coherent states η_z^F are now

$$\eta_z^F = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n^F \tag{4.41}$$

which, as vectors in $\mathfrak{H}_F$, are well defined and normalized for all $z \in \mathbb{C}$. However, when considered as vectors in $\mathfrak{H}$ and rewritten as

$$\eta_z^F = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n \phi_n}{f(n)!\sqrt{n!}} \tag{4.42}$$

they are no longer normalized and defined only on the domain (see (4.35) and (4.36)),

$$\mathcal{D} = \left\{ z \in \mathbb{C} \,\middle|\, |z| < L = \frac{1}{\rho} \right\}. \tag{4.43}$$

The operators a_F and $a_F^\dagger$ act on the vectors ϕ_n^F as

$$a_F \phi_n^F = \sqrt{n}\phi_{n-1}^F \qquad a_F^\dagger \phi_n^F = \sqrt{n+1}\phi_{n+1}^F. \tag{4.44}$$

The operator $A = a_F$, considered as an operator on $\mathfrak{H}$, and its adjoint $A^\dagger$ on $\mathfrak{H}$ act on the original basis vectors ϕ_n in the manner

$$A\phi_n = f(n)\sqrt{n}\phi_{n-1} \qquad A^\dagger\phi_n = f(n+1)\sqrt{n+1}\phi_{n+1} \tag{4.45}$$

and thus, we may write, in an obvious notation,

$$A = af(N) \qquad A^\dagger = f(N)a^\dagger \tag{4.46}$$

as operators on $\mathfrak{H}$.

Thus, up to normalization, the CS defined in (4.42) are the well-known *nonlinear coherent states* of quantum optics (Man'ko *et al* 1997).

As a specific physical example of such a family of coherent states, we might mention the function $f(n) = L_n^{(0)}(\eta^2)\big[(n+1)L_n^{(0)}(\eta^2)\big]^{-1}$, where $L_n^m(x)$ are generalized Laguerre polynomials and η is the so-called Lamb–Dicke parameter. These states appear as the stationary states of the centre of mass motion of a trapped and bichromatically laser driven ion, far from the Lamb–Dicke regime (Filho *et al* 1996).

The dual coherent states $\eta_z^{F^{-1}}$, as vectors in the Hilbert space $\mathfrak{H}_{F^{-1}}$, will be well-defined vectors in $\mathfrak{H}$ only if

$$\lim_{n\to\infty} \left[\frac{t(n+1)}{t(n)}\right]^2 \frac{1}{n+1} = \widetilde{\rho} < \infty. \tag{4.47}$$

In this case we have

$$\eta_z^{F^{-1}} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{f(n)!z^n}{\sqrt{n!}}\phi_n \tag{4.48}$$

and they are defined (as vectors in $\mathfrak{H}$) on the domain

$$\widetilde{\mathcal{D}} = \left\{ z \in \mathbb{C} \,\middle|\, |z| < \widetilde{L} = \frac{1}{\sqrt{\widetilde{\rho}}} \right\}. \tag{4.49}$$

Equations (4.48) and (4.49) should be compared to (4.42) and (4.43). We also have

$$\langle \eta_z^{F^{-1}} | \eta_z^F \rangle_{\mathfrak{H}} = 1 \tag{4.50}$$

for all $z \in \mathcal{D} \cap \widetilde{\mathcal{D}}$.

A resolution of the identity of $\mathfrak{H}$ can be obtained in terms of the vectors η_z^F (or $\eta_z^{F^{-1}}$) by solving a moment problem. Thus, for example, for the vectors (4.42) to satisfy

$$\iint_{\mathcal{D}} |\eta_z^F\rangle\langle\eta_z^F| \mathcal{N}(|z|^2)\, \mathrm{d}\mu(z,\bar{z}) = I \tag{4.51}$$

where $\mathrm{d}\mu(z,\bar{z}) = \mathrm{d}\lambda(r)\,\mathrm{d}\theta$ $(z = r\,\mathrm{e}^{\mathrm{i}\theta})$, the measure $\mathrm{d}\lambda$ must satisfy the moment conditions

$$\int_0^L r^{2n}\,\mathrm{d}\lambda(r) = \frac{[f(n)!]^2\, n!}{2\pi} \qquad n = 0, 1, 2, \ldots. \tag{4.52}$$

As is well known, the most nonclassical features of nonlinear coherent states lie in their squeezing, antibunching and sub-Poissonian properties, which all depend crucially on the choice of the nonlinearity function. These properties have been studied for nonlinear coherent states of the dual type (4.48) in Roy and Roy (2000).

A highly instructive example of the duality between families of nonlinear coherent states is provided by the Gilmore–Perelomov (Gilmore 1974) and Barut–Girardello (Barut and Girardello 1971) coherent states, defined for the discrete series representations of the group $SU(1,1)$. The Gilmore–Perelomov coherent states can be defined on $\mathfrak{H}$ as

$$\eta_z^{\mathrm{GP}} = \mathcal{N}_{\mathrm{GP}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \sqrt{\frac{(2\kappa+n-1)!}{n!}} z^n \phi_n \tag{4.53}$$

where $\mathcal{N}_{\mathrm{GP}}$ is a normalization factor, chosen so that $\left\|\eta_z^{\mathrm{GP}}\right\|_{\mathfrak{H}}^2 = 1$, and the parameter $\kappa = 1, 3/2, 2, 5/2, \ldots,$ labels the $SU(1,1)$ representation being used. These coherent states are defined on the open unit disc, $|z| < 1$. The Barut–Girardello coherent states, on the other hand, can be defined (again on $\mathfrak{H}$) as the vectors

$$\eta_z^{\mathrm{BG}} = \mathcal{N}_{\mathrm{BG}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(2\kappa+n-1)!}} \phi_n \qquad z \in \mathbb{C} \tag{4.54}$$

where, once more, $\mathcal{N}_{\mathrm{BG}}$ is chosen so that $\left\|\eta_z^{\mathrm{BG}}\right\|^2 = 1$. It is now immediately clear that the operator

$$T(N) = \sum_{n=0} \frac{1}{\sqrt{(2\kappa+n-1)!}} |\phi_n\rangle\langle\phi_n| \tag{4.55}$$

acts in the manner

$$\eta_z^{\mathrm{BG}} = \lambda_1 T(N) \eta_z \qquad \text{and} \qquad \eta_z^{\mathrm{GP}} = \lambda_2 T(N)^{-1} \eta_z \tag{4.56}$$

where λ_1 and λ_2 are constants, thus demonstrating the relation of duality between the two sets of coherent states.

A large class of dual pairs of the above type can be constructed by starting with the hypergeometric function,

$${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{x^n}{n!} \tag{4.57}$$

where the α_i and β_i are positive real numbers, q is an arbitrary positive integer and p is restricted by $q - 1 \leqslant p \leqslant q + 1$. (Here $(\gamma)_n$ is the usual Pochhammer symbol, $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1) = \Gamma(\gamma+n)/\Gamma(\gamma)$.) This series converges for all $x \in \mathbb{R}$ if $p = q$ and for all $|x| < 1$ if $p = q+1$. Then, going back to the canonical coherent states on $\mathfrak{H}$, we apply to them the operators

$$\begin{aligned} T &:= T(N) = \sum_{n=0}^{\infty} \left[\frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n}\right]^{\frac{1}{2}} |\phi_n\rangle\langle\phi_n| \\ T^{-1} &:= T(N)^{-1} = \sum_{n=0}^{\infty} \left[\frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n}\right]^{-\frac{1}{2}} |\phi_n\rangle\langle\phi_n|. \end{aligned} \tag{4.58}$$

It is then evident that the corresponding families of coherent states $\{\eta_z^F\}$ and $\{\eta_z^{F^{-1}}\}$ will be in duality. (Actually, it may be necessary to impose additional restrictions on the α_i and β_i, in order to ensure that the coherent states $\{\eta_z^F\}$ and $\{\eta_z^{F^{-1}}\}$, when defined on $\mathfrak{H}$, satisfy a resolution of the identity (Appl and Schiller 2003).)

To conclude this example, we note that from the manner in which the operators T and T^{-1} are defined, for the rescaled basis states (see (4.34) and (4.37)), we can always arrange to be in one of the following two situations:

(1) both T and T^{-1} are bounded;
(2) T is unbounded but T^{-1} is bounded.

In both cases, (2.4) holds, so that we always have a Gelfand triple.

4.3. Example 3. Squeezed bases

Our next example involves the use of *squeezed states* and squeezed bases (see, for example, Ali *et al* 2000, Simon *et al* 1988). Consider the symplectic group, $Sp(2, \mathbb{R})$, consisting of 2×2 real matrices M satisfying

$$M\beta M^T = \beta \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.59}$$

(Note that these matrices can also be characterized by the simple condition, $\det M = 1$, i.e., $Sp(2, \mathbb{R})$ is identical with the group $SL(2, \mathbb{R})$, of 2×2 real matrices of determinant one.) An element $M \in Sp(2, \mathbb{R})$ has the well-known decomposition (Sugiura 1990)

$$M = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} u^{-\frac{1}{2}} & 0 \\ 0 & u^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.60}$$

with $v \in \mathbb{R}$, $u > 0$, $0 < \theta \leqslant 2\pi$. We shall also write

$$M(u, v) = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} u^{-\frac{1}{2}} & 0 \\ 0 & u^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} u^{-\frac{1}{2}} & 0 \\ -vu^{-\frac{1}{2}} & u^{\frac{1}{2}} \end{pmatrix}. \tag{4.61}$$

Next, writing $z = \frac{1}{\sqrt{2}}(q - \mathrm{i}p)$, we introduce the vector $\mathbf{x}$ and the vector operator $\mathfrak{X}$:

$$\mathbf{x} = \begin{pmatrix} q \\ p \end{pmatrix} \qquad \mathfrak{X} = \begin{pmatrix} Q \\ P \end{pmatrix} \tag{4.62}$$

where Q and P are the position and momentum operators defined in (4.27) and (4.32), respectively. In terms of these quantities the canonical coherent states (1.2) can be rewritten as

$$\eta_{\mathbf{x}} := \eta_z = U(\mathbf{x})\phi_0 \qquad \text{where} \qquad U(\mathbf{x}) = \exp[-\mathrm{i}\mathbf{x}^T\beta\mathfrak{X}] \tag{4.63}$$

and $U(\mathbf{x})$ is unitary on $\mathfrak{H}$. If $\binom{Q'}{P'} = M\mathfrak{X}$, $M \in Sp(2, \mathbb{R})$, then since $[Q', P'] = [Q, P] = \mathrm{i}I$, there exists a unitary operator $U(M)$ on $\mathfrak{H}$ such that (with a slight abuse of notation)

$$U(M)\mathfrak{X}U(M)^\dagger = M^{-1}\mathfrak{X} \qquad \text{and} \qquad U(M)U(\mathbf{x})U(M)^\dagger = U(M\mathbf{x}). \tag{4.64}$$

Taking $\mathfrak{H} = L^2(\mathbb{R}, \mathrm{d}x)$ and $\phi_0 = \pi^{-\frac{1}{4}}\,\mathrm{e}^{-\frac{x^2}{2}}$, the states

$$\begin{aligned} \eta_{\mathbf{x}}^{u,v} &= U(\mathbf{x})U(M(u, v))\phi_0 \\ \left(\eta_{\mathbf{x}}^{u,v}\right)(x) &= \left[\frac{u}{\pi}\right]^{\frac{1}{4}} \exp\left(\mathrm{i}\left(x - \frac{q}{2}\right)p\right) \exp\left(-\frac{1}{2}(x-q)(u+\mathrm{i}v)(x-q)\right) \end{aligned} \tag{4.65}$$

are generalized Gaussians and for $v = 0$, $u = \frac{1}{s^2}$ these are *squeezed states*.

For fixed $M(u, v) \in Sp(2, \mathbb{R})$, let $T^{-1} = U(M(u, v))$ and set $\phi_n^{u,v} = \phi_n^F = U(M(u, v))\phi_n$. We call the resulting basis a *squeezed basis*. Then

$$\eta_z^F = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n^{u,v} \tag{4.66}$$

and since by (4.64),

$$U(M(u,v))U(\mathbf{x}) = U(M(u,v))U(\mathbf{x})U(M(u,v))^\dagger U(M(u,v)) = U(M\mathbf{x})U(M(u,v))$$

we obtain

$$\eta^F_z = \eta^{u,v}_{\mathbf{x}'} \qquad \text{where} \quad \mathbf{x}' = M(u,v)\mathbf{x}. \tag{4.67}$$

Thus, squeezing the basis results in squeezing the coherent states. The dual family of coherent states consists of the vectors $\eta^{1/u,-v}_{\mathbf{x}''}$, with $\mathbf{x}'' = M(1/u,-v)\mathbf{x}$. Since $U(M(u,v))$ is unitary on $\mathfrak{H}$, the algebra generated by the operators A and $A^\dagger$ is the same as that generated by a and $a^\dagger$.

5. Some operator algebras

In this section we take a closer look at the two sets of operators $a_F, a^\dagger_F$ and $a_{F^{-1}}, a^\dagger_{F^{-1}}$ and the algebras generated by them (under commutation), in the special case when the operators T and F have the forms given in (4.37). Note that both T and F are positive operators. As noted earlier, on the Hilbert space $\mathfrak{H}_F$ the operators $a_F, a^\dagger_F$ are adjoints of each other and satisfy the commutation relation $\left[a_F, a^\dagger_F\right] = I$, while on the Hilbert space $\mathfrak{H}_{F^{-1}}$ the operators $a_{F^{-1}}, a^\dagger_{F^{-1}}$ are mutual adjoints, satisfying $\left[a_{F^{-1}}, a^\dagger_{F^{-1}}\right] = I$. As before, let us write $A = a_F$, when this operator acts on $\mathfrak{H}$ and similarly we write $A' = a_{F^{-1}}$ to denote the action of $a_{F^{-1}}$ on $\mathfrak{H}$. Since $a_F = T^{-1}aT$ and $a^\dagger_{F^{-1}} = Ta^\dagger T^{-1}$ and since T and T^{-1} are positive operators, we have for the adjoint of A on $\mathfrak{H}$,

$$A^\dagger = Ta^\dagger T^{-1} = a^\dagger_{F^{-1}} \tag{5.1}$$

and similarly, for the adjoint of A' on $\mathfrak{H}$ we have

$$A'^\dagger = T^{-1}a^\dagger T = a^\dagger_F. \tag{5.2}$$

Moreover (see (4.46)),

$$A = af(N) \qquad A^\dagger = f(N)a^\dagger \qquad \text{and} \qquad A' = af(N)^{-1} \qquad A'^\dagger = f(N)^{-1}a^\dagger \tag{5.3}$$

with

$$[A, A'^\dagger] = [A', A^\dagger] = I. \tag{5.4}$$

In addition, we have the four other easily verifiable commutation relations,

$$\begin{aligned} [A, A^\dagger] &= f(N+1)^2(N+1) - f(N)^2 N \\ [A', A'^\dagger] &= f(N+1)^{-2}(N+1) - f(N)^{-2}N \\ [A, A'] &= a^2[f(N-1)f(N)^{-1} - f(N-1)^{-1}f(N) \\ [A^\dagger, A'^\dagger] &= [f(N)f(N-1)^{-1} - f(N)^{-1}f(N-1)]a^{\dagger 2}. \end{aligned} \tag{5.5}$$

Consider now the *displacement operators* on $\mathfrak{H}$,

$$D(z) = \mathrm{e}^{za^\dagger - \bar{z}a} = U(\mathbf{x}) \qquad z \in \mathbb{C}. \tag{5.6}$$

These operators are unitary on $\mathfrak{H}$ and in view of the relation

$$D(z_1)D(z_2) = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\bar{z}_1 z_2)}D(z_1 + z_2) \tag{5.7}$$

together they realize a unitary projective representation of the Weyl–Heisenberg group on $\mathfrak{H}$. Moreover,

$$\eta_z = D(z)\phi_0 = \mathrm{e}^{-\frac{|z|^2}{2}}\,\mathrm{e}^{za^\dagger}\phi_0. \tag{5.8}$$

The unitary images of $D(z)$ on $\mathfrak{H}_F$ and $\mathfrak{H}_{F^{-1}}$ are

$$D_F(z) = T^{-1}D(z)T = \mathrm{e}^{za_F^\dagger - \bar{z}a_F} \qquad \text{and} \qquad D_{F^{-1}}(z) = TD(z)T^{-1} = \mathrm{e}^{za_{F^{-1}}^\dagger - \bar{z}a_{F^{-1}}} \tag{5.9}$$

respectively, again defined for all $z \in \mathbb{C}$ and realizing unitary projective representations of the Weyl–Heisenberg group on $\mathfrak{H}_F$ and $\mathfrak{H}_{F^{-1}}$, respectively. Also, just as in (5.8), we have

$$\eta_z^F = D_F(z)\phi_0 = \mathrm{e}^{-\frac{|z|^2}{2}}\,\mathrm{e}^{za_F^\dagger}\phi_0 \qquad \eta_z^{F^{-1}} = D_{F^{-1}}(z)\phi_0 = \mathrm{e}^{-\frac{|z|^2}{2}}\,\mathrm{e}^{za_{F^{-1}}^\dagger}\phi_0. \tag{5.10}$$

Letting them act on $\mathfrak{H}$, we write $V(z)$ and $V'(z)$ for these two operators, so that using (5.1) and (5.2), we have

$$V(z) := D_F(z) = \mathrm{e}^{zA'^\dagger - \bar{z}A} \qquad \text{and} \qquad V'(z) := D_{F^{-1}}(z) = \mathrm{e}^{zA^\dagger - \bar{z}A'} \tag{5.11}$$

operators which have been studied in Roy and Roy (2000). Thus, as operators on $\mathfrak{H}$,

$$V'(z) = V(-z)^\dagger = [V(z)^{-1}]^\dagger. \tag{5.12}$$

However, on $\mathfrak{H}$ the operator $V(z)$ is only defined for $z \in \mathcal{D}$, where $\mathcal{D}$ is the domain (4.43), while $V'(z)$ is defined for $z \in \widetilde{\mathcal{D}}$ (see (4.49)), so that (5.12) only holds on $\mathcal{D} \cap \widetilde{\mathcal{D}}$. Also, if $z_1, z_2, z_1 + z_2 \in \mathcal{D}$ then we have a relation similar to (5.7) for $V(z)$:

$$V(z_1)V(z_2) = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\bar{z}_1 z_2)}V(z_1 + z_2). \tag{5.13}$$

Similarly, if $z_1, z_2, z_1 + z_2 \in \widetilde{\mathcal{D}}$ then we have for $V'(z)$ the analogous relation

$$V'(z_1)V'(z_2) = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\bar{z}_1 z_2)}V'(z_1 + z_2). \tag{5.14}$$

Thus, if $\mathcal{D} = \mathbb{C}$ (respectively, $\widetilde{\mathcal{D}} = \mathbb{C}$) then the operators $V(z)$ (respectively, $V'(z)$) define a *non-unitary* projective representation of the Weyl–Heisenberg group on $\mathfrak{H}$. In the case where $\mathcal{D} = \widetilde{\mathcal{D}} = \mathbb{C}$, then both $V(z)$ and $V'(z)$ realize non-unitary representations of the Weyl–Heisenberg group on $\mathfrak{H}$ and (5.12) implies that these representations are *contragredient* to each other. This could happen, if for example, both T and T^{-1} are bounded operators. Another possibility could be when T and T^{-1} have the forms

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_p)_n}|\phi_n\rangle\langle\phi_n| \\ T^{-1} &= \sum_{n=0}^{\infty} \left[\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_p)_n}\right]^{-1}|\phi_n\rangle\langle\phi_n| \end{aligned} \tag{5.15}$$

for real numbers α_j and β_j. (This corresponds to taking $p = q$ in (4.57).) But in all cases, one member of a dual pair gives rise to a non-unitary projective representation of the Weyl–Heisenberg group. In other words, each dual pair of nonlinear coherent states is characterized by such a representation.

6. Discussion

Let us make two final comments before ending this paper. First, we note that the general method which emerges for constructing nonlinear coherent states is to take the two operators T, $D(z)$, defined as in (4.37) and (5.6), a *fiducial vector* ϕ_0, and then to write

$$\eta_z^{\mathrm{nl}} = T^{-1}D(z)\phi_0. \tag{6.1}$$

The set of values of z for which these vectors are defined then depends on T. The dual family of nonlinear CS is defined by replacing T^{-1} by T. The canonical CS form a self-dual family. The second, and related comment is on whether one could make contact with the type of coherent states, associated with Hamiltonians with discrete spectra, which were constructed in Gazeau and Klauder (1999). These latter coherent states, which we shall call Gazeau–Klauder coherent states, are parametrized by action and angle variables J, γ and are of the type

$$|J, \gamma\rangle = \mathcal{N}(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{J^{\frac{n}{2}} \exp[-\mathrm{i}\varepsilon_n \gamma]}{\sqrt{\varepsilon_n!}} \phi_n \tag{6.2}$$

where $J \geqslant 0, -\infty < \gamma < \infty$ and $\varepsilon_n > 0, n = 1, 2, 3, \ldots, \varepsilon_0 = 0$, are the eigenvalues (non-degenerate) of some Hamiltonian H, corresponding to the eigenvectors ϕ_n. Although it is possible to find a transformation operator T mapping the canonical coherent states (1.2)—which can also be considered to be of the Gazeau–Klauder type—to states of the type (6.2), such an operator would not be of the type considered in this paper, because these operators lead to the states (3.3), labelled by a complex variable w. The scalar product, $\langle\psi|\zeta_w\rangle$, with respect to an arbitrary vector $\psi \in \mathfrak{H}$ defines, up to a factor $\mathcal{N}'(|w|^2)^{-\frac{1}{2}}\Omega(w)$, an analytic function of w over some domain $\mathcal{D}$. This would not be the case if we replaced ζ_w by $|J, \gamma\rangle$ in this scalar product, except when $\varepsilon_n = n$ for all n. However, it is interesting to study such transformations generically and we propose to do so in a future publication.

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